

ON AN IDENTITY FOR H-FUNCTION

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ABSTRACT. The main objective of this research note is to provide an identity for the H-function, which generalizes two identities involving H-function obtained earlier by Rathie and Rathie et al.

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1. INTRODUCTION

In 1981, Rathie [2] established the following identity for the H-function viz.

$$\begin{aligned} H_{p+1, q+1}^{m, n} \left[z \middle| \begin{matrix} {}_1(a_j, e_j)_p, (\alpha, \lambda) \\ {}_1(b_j, f_j)_q, (\alpha, \lambda) \end{matrix} \right] \\ = \frac{1}{2\pi i} \left\{ e^{i\pi\alpha} H_{p, q}^{m, n} \left[ze^{-i\pi\lambda} \middle| \begin{matrix} {}_1(a_j, e_j)_p \\ {}_1(b_j, f_j)_q \end{matrix} \right] \right. \\ \left. - e^{-i\pi\alpha} H_{p, q}^{m, n} \left[ze^{i\pi\lambda} \middle| \begin{matrix} {}_1(a_j, e_j)_p \\ {}_1(b_j, f_j)_q \end{matrix} \right] \right\} \end{aligned} \quad (1.1)$$

Very recently, Rathie et al.[3] established another identity for the H-function viz.

$$\begin{aligned} H_{p+1, q+1}^{m+1, n+1} \left[z \middle| \begin{matrix} (\alpha, \lambda), {}_1(a_j, e_j)_p \\ (\alpha, \lambda), {}_1(b_j, f_j)_q \end{matrix} \right] \\ = e^{i\pi\alpha} H_{p+1, q+1}^{m+1, n+1} \left[ze^{-i\pi\lambda} \middle| \begin{matrix} (2\alpha, 2\lambda), {}_1(a_j, e_j)_p \\ (2\alpha, 2\lambda), {}_1(b_j, f_j)_q \end{matrix} \right] \\ + e^{-i\pi\alpha} H_{p+1, q+1}^{m+1, n+1} \left[ze^{i\pi\lambda} \middle| \begin{matrix} (2\alpha, 2\lambda), {}_1(a_j, e_j)_p \\ (2\alpha, 2\lambda), {}_1(b_j, f_j)_q \end{matrix} \right] \end{aligned} \quad (1.2)$$

For interesting applications of the identities (1.1) and (1.2), we refer the recent paper by Rathie et al.[3]

The aim of this short note is to provide a natural generalization of (1.1) and (1.2).

2. MAIN RESULT

The identity for the H-function to be established in this note is the following.

$$\begin{aligned}
& 2\pi i H_{p+2, q+2}^{m+1, n+1} \left[z \mid \begin{matrix} (\beta, \delta), {}_1(a_j, e_j)_p, (\alpha, \lambda) \\ (\beta, \delta), {}_1(b_j, f_j)_q, (\alpha, \lambda) \end{matrix} \right] \\
&= e^{i\pi(\alpha+\beta)} H_{p+1, q+1}^{m+1, n+1} \left[ze^{-i\pi(\lambda+\delta)} \mid \begin{matrix} (2\beta, 2\delta), {}_1(a_j, e_j)_p \\ (2\beta, 2\delta), {}_1(b_j, f_j)_q \end{matrix} \right] \\
&+ e^{i\pi(\alpha-\beta)} H_{p+1, q+1}^{m+1, n+1} \left[ze^{-i\pi(\lambda-\delta)} \mid \begin{matrix} (2\beta, 2\delta), {}_1(a_j, e_j)_p \\ (2\beta, 2\delta), {}_1(b_j, f_j)_q \end{matrix} \right] \\
&- e^{-i\pi(\alpha-\beta)} H_{p+1, q+1}^{m+1, n+1} \left[ze^{i\pi(\lambda-\delta)} \mid \begin{matrix} (2\beta, 2\delta), {}_1(a_j, e_j)_p \\ (2\beta, 2\delta), {}_1(b_j, f_j)_q \end{matrix} \right] \\
&- e^{-i\pi(\alpha+\beta)} H_{p+1, q+1}^{m+1, n+1} \left[ze^{i\pi(\lambda+\delta)} \mid \begin{matrix} (2\beta, 2\delta), {}_1(a_j, e_j)_p \\ (2\beta, 2\delta), {}_1(b_j, f_j)_q \end{matrix} \right] \tag{2.1}
\end{aligned}$$

where $H_{p, q}^{m, n}[z]$ is the well known H-function[1].

Proof : In order to establish the identity (2.1), we proceed as follows.

Denoting the left-hand of H-function by I, expressing the H-function with the help of its definition we have,

$$I = 2\pi i \frac{1}{2\pi i} \int_L \theta(s) z^s \frac{\Gamma(\beta - \delta s) \Gamma(1 - \beta + \delta s)}{\Gamma(\alpha - \lambda s) \Gamma(1 - \alpha + \lambda s)} ds \tag{2.2}$$

where $\theta(s)$ is given by

$$\theta(s) = \frac{\prod_{j=1}^m \Gamma(b_j - f_j s) \prod_{j=1}^n \Gamma(1 - a_j + e_j s)}{\prod_{j=m+1}^q \Gamma(1 - b_j + f_j s) \prod_{j=n+1}^p \Gamma(a_j - e_j s)} \tag{2.3}$$

Using the results

$$\Gamma(\beta - \delta s) \Gamma(1 - \beta + \delta s) = 2\pi \frac{\Gamma(2\beta - 2\delta s) \Gamma(1 - 2\beta + 2\delta s)}{\Gamma(\frac{1}{2} + \beta - \delta s) \Gamma(\frac{1}{2} - \beta + \delta s)} \tag{2.4}$$

$$\sin \pi z = \frac{\pi}{\Gamma(z) \Gamma(1-z)} = \frac{e^{i\pi z} - e^{-i\pi z}}{2i} \tag{2.5}$$

and

$$\cos \pi z = \frac{\pi}{\Gamma(\frac{1}{2} - z) \Gamma(\frac{1}{2} + z)} = \frac{e^{i\pi z} + e^{-i\pi z}}{2} \tag{2.6}$$

and after some algebra, we have

$$\begin{aligned}
I &= \frac{1}{2\pi i} \int_L \theta(s) z^s \Gamma(2\beta - 2\delta s) \Gamma(1 - 2\beta + 2\delta s) \\
&\quad \cdot \left(e^{i\pi(\alpha-\lambda s)} - e^{-i\pi(\alpha-\lambda s)} \right) \left(e^{i\pi(\beta-\delta s)} + e^{-i\pi(\beta-\delta s)} \right) ds \\
&= \frac{1}{2\pi i} \int_L \theta(s) z^s \Gamma(2\beta - 2\delta s) \Gamma(1 - 2\beta + 2\delta s) \\
&\quad \cdot \left\{ e^{i\pi(\alpha+\beta-\lambda s-\delta s)} + e^{i\pi(\alpha-\beta-\lambda s+\delta s)} \right. \\
&\quad \left. - e^{-i\pi(\alpha-\beta-\lambda s+\delta s)} - e^{-i\pi(\alpha+\beta-\lambda s-\delta s)} \right\} ds \tag{2.7}
\end{aligned}$$

Now, breaking in to four parts and after some simplification, using the definition of H-function, we easily arrive at the right-hand side of (2.1).

This completes the proof of the identity (2.1).

3. SPECIAL CASES

- (a) In (2.1), if we take $\delta = 0$, we get, after some simplification, the identity obtained earlier by Rathie[2].
- (b) In (2.1), if we take $\lambda = 0$, we get, after some simplification, the identity obtained very recently by Rathie et al.[3].

4. CONCLUDING REMARKS

If we use the result (2.4), we get the following identity for the H-function.

$$\begin{aligned}
 & H_{p+2, q+2}^{m+1, n+1} \left[z \mid \begin{matrix} (\beta, \delta), {}_1(a_j, e_j)_p, (\alpha, \lambda) \\ (\beta, \delta), {}_1(b_j, f_j)_q, (\alpha, \lambda) \end{matrix} \right] \\
 &= H_{p+4, q+4}^{m+2, n+2} \left[z \mid \begin{matrix} (2\beta, 2\delta), \left(\frac{1}{2} + \alpha, \lambda\right), {}_1(a_j, e_j)_p, (2\alpha, 2\lambda), \left(\frac{1}{2} + \beta, \delta\right) \\ (2\beta, 2\delta), \left(\frac{1}{2} + \alpha, \lambda\right), {}_1(b_j, f_j)_q, (2\alpha, 2\lambda), \left(\frac{1}{2} + \beta, \delta\right) \end{matrix} \right] \quad (4.1)
 \end{aligned}$$

Further in this, if we take $\delta = 0$ and $\lambda = 0$, we respectively get

$$\begin{aligned}
 & H_{p+1, q+1}^{m, n} \left[z \mid \begin{matrix} {}_1(a_j, e_j)_p, (\alpha, \lambda) \\ {}_1(b_j, f_j)_q, (\alpha, \lambda) \end{matrix} \right] \\
 &= \frac{1}{2\pi} H_{p+2, q+2}^{m+1, n+1} \left[z \mid \begin{matrix} \left(\frac{1}{2} + \alpha, \lambda\right), {}_1(a_j, e_j)_p, (2\alpha, 2\lambda) \\ \left(\frac{1}{2} + \alpha, \lambda\right), {}_1(b_j, f_j)_q, (2\alpha, 2\lambda) \end{matrix} \right] \quad (4.2)
 \end{aligned}$$

and

$$\begin{aligned}
 & H_{p+1, q+1}^{m, n} \left[z \mid \begin{matrix} (\beta, \delta), {}_1(a_j, e_j)_p \\ (\beta, \delta), {}_1(b_j, f_j)_q \end{matrix} \right] \\
 &= 2\pi H_{p+2, q+2}^{m+1, n+1} \left[z \mid \begin{matrix} (2\beta, 2\delta), {}_1(a_j, e_j)_p, \left(\frac{1}{2} + \beta, \delta\right) \\ (2\beta, 2\delta), {}_1(b_j, f_j)_q, \left(\frac{1}{2} + \beta, \delta\right) \end{matrix} \right] \quad (4.3)
 \end{aligned}$$

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